Polynomial Functions

Roots, Multiplicity, the Gauss–Lucas Theorem (statement), and Taylor Polynomials

Lecture Notes

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1 Polynomials Are Entire

A **polynomial function** is

$$f(z) = \sum_{n=0}^{d} a_n z^n.$$

By the differentiation rules for products and sums, $f'(z) = \sum_{n=1}^{d} n a_n z^{n-1}$, so f is **entire**: analytic at every $z \in \mathbb{C}$.

Example. $p(z) = z^2 = (x^2 - y^2) + 2ixy$, where $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$ – the real and imaginary parts of z^2 recovered from its expression in $z, \bar{z}$.

Proposition (which polynomials in x, y are analytic)

Let $F(x,y) = \sum_{j,k\geq 0} c_{jk}\, x^j y^k$ be a polynomial in the real variables x, y with complex coefficients, and define $f(z) := F(x,y)$ for $z = x + iy$. Then f is analytic on $\mathbb{C}$ if and only if, after substituting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and expanding, f involves no $\bar{z}$ – i.e. $f(z)$ is itself a polynomial in z alone.

Proof. ($\Leftarrow$) If $f(z) = p(z)$ is a polynomial in z alone, it is entire by the computation above.

($\Rightarrow$) The substitution $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ is an invertible linear change of coordinates, so expanding F produces a unique representation $f(z) = \sum_{m,n\geq 0} a_{mn}\, z^m \bar{z}^n$. Since F is a real polynomial, f is automatically real-differentiable everywhere; if f is also analytic, the Cauchy–Riemann criterion $\partial f/\partial \bar{z} \equiv 0$ gives

$$0 \equiv \frac{\partial f}{\partial \bar{z}} = \sum_{m\geq 0,\, n\geq 1} n\, a_{mn}\, z^m \bar{z}^{n-1},$$

and since the representation in $z, \bar{z}$ is unique, every coefficient a_{mn} with $n \geq 1$ must vanish. Hence $f(z) = \sum_m a_{m0} z^m$ is a polynomial in z alone. □

2 Roots and Multiplicity

Fundamental Theorem of Algebra (stated; proof deferred)

If p is a polynomial of degree $d \geq 1$, then p has a root (zero) z_1: $p(z_1) = 0$.

Dividing p by $(z - z_1)$: $p(z) = q(z)(z - z_1)$, where the remainder is the constant $p(z_1) = 0$. The quotient q has degree $d - 1$, so it too has a root, and by induction

$$p(z) = a_d (z - z_1) \cdots (z - z_d).$$

Some of the roots $z_1, \dots, z_d$ may coincide.

Definition (multiplicity)

The **multiplicity** (or **order**) of z_0 as a root of p is $\#\{j : z_j = z_0\}$. Equivalently, $p(z) = (z-z_0)^h q(z)$ with $q(z_0) \neq 0$: h is the order. This is equivalent to

$$p(z_0) = p'(z_0) = \cdots = p^{(h-1)}(z_0) = 0, \qquad p^{(h)}(z_0) \neq 0.$$

Lemma. If $p(z) = (z - z_0)g(z)$ for a polynomial g, then $p^{(k)}(z_0) = k\, g^{(k-1)}(z_0)$ for every $k \geq 1$.

Proof. Induction on k. For $k = 1$, the product rule gives $p'(z) = g(z) + (z - z_0)g'(z)$, so $p'(z_0) = g(z_0)$. Assuming $p^{(k)}(z) = k\, g^{(k-1)}(z) + (z - z_0)g^{(k)}(z)$, differentiating once more gives

$$p^{(k+1)}(z) = k\, g^{(k)}(z) + g^{(k)}(z) + (z - z_0)g^{(k+1)}(z) = (k + 1)\, g^{(k)}(z) + (z - z_0)g^{(k+1)}(z),$$

the same identity with $k + 1$ in place of k. Evaluating at z_0 at each step gives $p^{(k)}(z_0) = k\, g^{(k-1)}(z_0)$. □

Proof of the equivalence, by induction on h. For $h = 0$, both sides say $p(z_0) \neq 0$ (taking $q = p$ on the left; the right side is $p^{(0)}(z_0) = p(z_0) \neq 0$, with no vanishing conditions), so the two are trivially equivalent.

Assume the equivalence holds for h; we prove it for $h + 1$.

($\Rightarrow$) Suppose $p(z) = (z - z_0)^{h+1}q(z)$, $q(z_0) \neq 0$. Write $p(z) = (z - z_0)g(z)$ with $g(z) := (z - z_0)^h q(z)$; since g itself satisfies the order-h hypothesis, the inductive hypothesis gives $g(z_0) = \cdots = g^{(h-1)}(z_0) = 0$ and $g^{(h)}(z_0) \neq 0$. By the Lemma, $p^{(k)}(z_0) = k\, g^{(k-1)}(z_0)$ for $k \geq 1$ (and $p(z_0) = 0$ directly, since p has the factor $z - z_0$); so $p^{(k)}(z_0) = 0$ for $1 \leq k \leq h$, while $p^{(h+1)}(z_0) = (h + 1)\, g^{(h)}(z_0) \neq 0$. Together with $p(z_0) = 0$, this is exactly the order-$(h + 1)$ derivative condition.

($\Leftarrow$) Suppose $p(z_0) = \cdots = p^{(h)}(z_0) = 0$ and $p^{(h+1)}(z_0) \neq 0$. Since $p(z_0) = 0$, $p(z) = (z - z_0)g(z)$ for the polynomial $g(z) := p(z)/(z - z_0)$. By the Lemma, $0 = p^{(k)}(z_0) = k\, g^{(k-1)}(z_0)$ for $1 \leq k \leq h$, so $g(z_0) = \cdots = g^{(h-1)}(z_0) = 0$; and $0 \neq p^{(h+1)}(z_0) = (h + 1)\, g^{(h)}(z_0)$, so $g^{(h)}(z_0) \neq 0$. By the inductive hypothesis applied to g, this means $g(z) = (z - z_0)^h q(z)$ with $q(z_0) \neq 0$, hence $p(z) = (z - z_0)^{h+1}q(z)$, $q(z_0) \neq 0$. □

Proposition

If z_0 is a root of p of order $h \geq 1$, then z_0 is a root of p' of order $h - 1$.

Proof. Write $p(z) = (z - z_0)^h q(z)$, $q(z_0) \neq 0$. Then

$$p'(z) = h(z - z_0)^{h-1}q(z) + (z - z_0)^h q'(z) = (z - z_0)^{h-1} \underbrace{[h\, q(z) + (z - z_0)q'(z)]}_{=:r(z)},$$

and $r(z_0) = h\, q(z_0) \neq 0$. So z_0 has order $h - 1$ as a root of p'. □

3 The Gauss–Lucas Theorem

Carl Friedrich Gauss
(1777–1855)

Édouard Lucas
(1842–1891)

Theorem (Gauss–Lucas)

Let $z_1, \dots, z_J$ be the (distinct) zeros of a polynomial p. Then all zeros of $p'(z)$ lie inside the convex hull of $\{z_1, \dots, z_J\}$.

Definition (convex hull)

The **convex hull** of $\{z_1, \dots, z_J\}$ is

$$H = \Big\{ w \in \mathbb{C} : \exists\, m_1, \dots, m_J \geq 0,\ \textstyle\sum_j m_j = 1,\ w = \sum_j m_j z_j \Big\},$$

geometrically, the smallest convex set containing $\{z_1, \dots, z_J\}$.

Corollary. For every $c \in \mathbb{C}$, the roots of $p'(z)$ lie in the convex hull of the roots of $p(z) = c$, since $(p(z) - c)' = p'(z)$.

4 Taylor Polynomials

Brook Taylor (1685–1731)

Theorem (Taylor polynomial)

Let p be a polynomial of degree d and $z_0 \in \mathbb{C}$. Then

$$p(z) = \sum_{k=0}^{d} \frac{p^{(k)}(z_0)}{k!} (z - z_0)^k.$$

Proof. Induction on d. For $d = 0$, $p(z) = c = p(z_0)$, and the formula holds trivially.

Inductive step. Since $p(z) - p(z_0)$ vanishes at z_0, it is divisible by $(z - z_0)$:

$$q(z) := \frac{p(z) - p(z_0)}{z - z_0}, \qquad p(z) = p(z_0) + (z - z_0)q(z),$$

where q is a polynomial of degree $d - 1$. By the Lemma above, $p^{(k)}(z_0) = k\, q^{(k-1)}(z_0)$ for $k \geq 1$, i.e., $q^{(j)}(z_0) = \dfrac{p^{(j+1)}(z_0)}{j+1}$ for $j \geq 0$.

By the inductive hypothesis applied to q (degree $d - 1$),

$$q(z) = \sum_{k=0}^{d-1} \frac{q^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{d-1} \frac{p^{(k+1)}(z_0)}{(k+1)!}(z - z_0)^k.$$

Therefore

$$p(z) = p(z_0) + (z - z_0) \sum_{k=0}^{d-1} \frac{p^{(k+1)}(z_0)}{(k+1)!}(z - z_0)^k = p(z_0) + \sum_{k=1}^{d} \frac{p^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{d} \frac{p^{(k)}(z_0)}{k!}(z - z_0)^k.$$

This completes the induction. □